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On the Proca and Wigner realizations for spin one

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Abstract. We show, by relating the Proca covariant amplitude and the corresponding canonical (Wigner) amplitude for massive elementary systems with spin one, how the transversality condition arises in a natural way. The connection between both realizations extends also to establish the form of the scalar product and the position operator in covariant space.

1. Introduction

It is well known how elementary quantum systems are characterized by the continuous unitary irreducible representations of $\tilde{\mathscr{P}}_0$, the universal covering group of the (connected) Poincaré or inhomogeneous Lorentz group (Wigner 1939). The rigorization of the Wigner procedure (and the extension to similar semi-direct-product groups) is due to Mackey (for a modern exposition see Simms 1968). The method also provides an explicit construction of the representation, namely, the so-called canonical or Wigner realization. However, one is usually more interested in the covariant realizations, as they are usually provided by the covariant equations.

Here we treat the case of $[m \neq 0, s = 1, \epsilon = +1]$ (mass, spin and sign of the energy) and compare the canonical realization with that provided by the Proca equation; in the following we recall briefly both cases: in § 2 we establish the connection between them (following mainly Pursey 1965) and in §§ 3 and 4 we obtain directly the form of the scalar product and the position operator.

For $(b, B) \in \tilde{\mathscr{P}}_0$, the Wigner representation for $[m \neq 0, 1, +]$ is given by

$$[U(b, B)\phi]^K(p) = e^{-ipb}[D_1\{q(p, B)\}]^K_l \phi^l[\Lambda_B^{-1}p] \tag{1.1}$$

with well known symbols; explicitly $pb = \boldsymbol{pb} - p^0b^0$, $q(p, B) = L^{-1}(p)BL(\Lambda_B^{-1}p)$ is an element of the little group of the point $\hat{p} = (0, 0, 0, m)$, $L(p): \hat{p} \rightsquigarrow p$ and D_1 is the three-dimensional representation of SU(2).

The (covariant) realization of the Proca equations uses the vectorial representation of the (homogeneous) Lorentz group and it is

$$[U(b, B)\psi]^\alpha(p) = e^{-ipb}[D_{\frac{1}{2}\frac{1}{2}}(B)]^\alpha_\beta \psi^\beta(\Lambda_B^{-1}p). \tag{1.2}$$

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If here $p \in \Omega_m^+$, the representation is not irreducible as $D_{\frac{1}{2}\frac{1}{2}} \to D_1 + D_0$ we must eliminate the superfluous $s = 0$ component (subsidiary or transversality condition in the Proca system).

2. Explicit relation between the vectorial and canonical realization

A general study for relating Wigner and covariant realizations has been made by Pursey (1965) (one can see also Niederer and O'Raifeartaigh 1970); in our case, if ψ^α is as in (1.2), we define

$$\phi^\alpha(p) = [D_{\frac{1}{2}\frac{1}{2}}\{L^{-1}(p)\}]^\alpha_\beta \psi^\beta(p) \tag{2.1}$$

and the induced representation from the law (1.2) is as in (1.1), except that $D_{\frac{1}{2}\frac{1}{2}}\{q(p, B)\}$ stands instead of $D_1\{q(p, B)\}$. We change the basis in the space of the $D_{\frac{1}{2}\frac{1}{2}}$ representation from the spinor formalism to the form

$$\psi^{(1}_1 = \psi^{\dot{1}}_1, \qquad \psi^{(1}_0 = \frac{1}{\sqrt{2}}\{\psi^{\dot{2}}_1 + \psi^{\dot{1}}_2\}, \qquad \psi^{(1}_{-1} = \psi^{\dot{2}}_2, \qquad \psi^{(0} = \frac{1}{\sqrt{2}}\{\psi^{\dot{2}}_1 - \psi^{\dot{1}}_2\} \tag{2.2}$$

which is adapted to the $D_1 + D_0$ splitting; from this and (2.1) it follows in particular that the elementarity condition ($s = 1$ only)

$$P_\mu A^\mu = \boldsymbol{p}\boldsymbol{A} - p^0 A^0 = 0 \tag{2.3}$$

that is, the traditional transversality condition, arises in a natural fashion in relating the covariant and the canonical amplitudes.

For these calculations, we have used the usual boost matrices

$$L(p) = [2m(p^0 + m)]^{-1/2}[(p^0 + m)I + \boldsymbol{\sigma} \cdot \boldsymbol{p}] \tag{2.4}$$

(see eg, Fonda and Ghirardi 1969; other choices are possible eg Niederer and O'Raifeartaigh 1970).

The explicit relations between the vectorial and canonical amplitudes are

$$\begin{bmatrix} \phi^{(1}_1 \\ \phi^{(1}_0 \\ \phi^{(1}_{-1} \\ \phi^{(0} \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{p_1(p_1 - \mathrm{i}p_2)}{m(p^0+m)} & -\mathrm{i} + \dfrac{p_2(p_1 - \mathrm{i}p_2)}{m(p^0+m)} & p_3\dfrac{p_1 - \mathrm{i}p_2}{m(p^0+m)} & -\dfrac{p_1 - \mathrm{i}p_2}{m} \\ \dfrac{-\sqrt{2}p_1p_3}{m(p^0+m)} & -\dfrac{\sqrt{2}p_2p_3}{m(p^0+m)} & -\sqrt{2}\left(1 + \dfrac{p_3^2}{m(p^0+m)}\right) & \sqrt{2}\dfrac{p_3}{m} \\ -\left(1 + \dfrac{p_1(p_1 + \mathrm{i}p_2)}{m(p^0+m)}\right) & -\mathrm{i} - \dfrac{p_2(p_1 + \mathrm{i}p_2)}{m(p^0+m)} & \dfrac{-p_3(p_1 + \mathrm{i}p_2)}{m(p^0+m)} & \dfrac{p_1 + \mathrm{i}p_2}{m} \\ \dfrac{\sqrt{2}p_1}{m} & \dfrac{\sqrt{2}p_2}{m} & \dfrac{\sqrt{2}p_3}{m} & -\dfrac{\sqrt{2}p^0}{m} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \\ A^0 \end{bmatrix} \tag{2.5}$$

$$
\begin{bmatrix} A^1 \\ A^2 \\ A^3 \\ A^0 \end{bmatrix}
= \begin{bmatrix}
\frac{1}{2}\left(1+\frac{p_1(p_1+\mathrm{i}p_2)}{m(p^0+m)}\right) & -\frac{1}{\sqrt{2}}\frac{p_1p_3}{m(p^0+m)} & -\frac{1}{2}\left(1+\frac{p_1(p_1-\mathrm{i}p_2)}{m(p^0+m)}\right) & -\frac{p_1}{\sqrt{2}m} \\
\frac{\mathrm{i}}{2}\left(1-\frac{\mathrm{i}p_2(p_1+\mathrm{i}p_2)}{m(p^0+m)}\right) & -\frac{1}{\sqrt{2}}\frac{p_2p_3}{m(p^0+m)} & \frac{\mathrm{i}}{2}\left(1+\frac{\mathrm{i}p_2(p_1-\mathrm{i}p_2)}{m(p^0+m)}\right) & -\frac{p_2}{\sqrt{2}m} \\
\frac{1}{2}\frac{p_3(p_1+\mathrm{i}p_2)}{m(p^0+m)} & -\frac{1}{\sqrt{2}}\left(1+\frac{p_3^2}{m(p^0+m)}\right) & -\frac{1}{2}\frac{p_3(p_1-\mathrm{i}p_2)}{m(p^0+m)} & \frac{-p_3}{\sqrt{2}m} \\
\frac{p_1+\mathrm{i}p_2}{2m} & -\frac{p_3}{\sqrt{2}m} & -\frac{p_1-\mathrm{i}p_2}{2m} & \frac{-p^0}{\sqrt{2}m}
\end{bmatrix}
\begin{bmatrix} \phi_1^{(1} \\ \phi_0^{(1} \\ \phi_{-1}^{(1} \\ \phi^{(0} \end{bmatrix}
\tag{2.6}
$$

3. The scalar product

Usually the scalar product is introduced in this representation by the consideration of $A^+A = A_\mu^* A^\mu$ as the unique invariant which we can make with the field A which we use for the description of the elementary system $[m, 1, +]$. We show that when the scalar product is correctly introduced, that is, it is transferred from the canonical realization, that one which has a natural definite scalar product, it has the former form.

From equations (2.5) we can calculate

$$|\phi_1^{(1}|^2+|\phi_0^{(1}|^2+|\phi_{-1}^{(1}|^2 = A^2 - A^{02}. \tag{3.1}$$

For the realization of these calculations, we can make the assumption that the amplitudes A^μ are real, which obviously is not true, because the amplitudes A^μ are in the support space of a representation of $\mathscr{P}_0$, and each translation is represented for a complex factor, (some phase). However, the four components A^μ are relatively real, say, they have the same phase, and hence all calculations lead to correct results, if the equation (3.1) is interpreted by

$$|\phi_1^{(1}|^2+|\phi_0^{(1}|^2+|\phi_{-1}^{(1}|^2 = A_\mu^* A^\mu.$$

As consequence of the transport of structure from the canonical realization, the scalar product has the expression

$$\langle A, A\rangle = \int_{\Omega_m^+} A^+(p)A(p)\,\mathrm{d}\mu(p).$$

4. The position operator

If the operator associated with any observable is defined in the canonical realization, one can obtain the corresponding operator in the Proca realization by means of the relations (2.5) and (2.6).

For example, we know the explicit form of the position operator in the minimal realization (Boya 1970, Fong and Rowe 1968) and we can define the position operator in the Proca realization by

$$x_{\mathrm{P}} = A x_{\mathrm{W}} A^{-1}$$

where A^{-1} and A are the matrices of the relations (2.5) and (2.6) respectively.

We can do the calculations for x_{P} as follows

$$x_{\mathrm{P}} = x_{\mathrm{W}} + A[x_{\mathrm{W}}, A^{-1}]$$

but it is known that

$$[\boldsymbol{x}_{\mathrm{W}}, A^{-1}] = \mathrm{i}\boldsymbol{\nabla}_{\mathrm{P}} A^{-1}$$

and a simple but troublesome calculation leads to the explicit expression for the position operator, which in reduced form, and simplified by making use of the transversality condition, is as follows (in $\hat{T}_4$ space)

$$x_j = \mathrm{i}\frac{\partial}{\partial p_j} - \frac{\mathrm{i}p_j}{2p_0^2} - \frac{p^\mu I_{\mu j}}{m(p^0+m)} - \frac{1}{p^0+m}\{I_{0j}+I_{j0}\} + \frac{p_j p_k}{m(p^0+m)}\{I_{0k}+I_{k0}\} - \frac{p_j^2}{m^2 p^0(p^0+m)}$$

where $I_{\mu\nu}$ is the matrix defined by

$$[I_{\mu\nu}]_{\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta}$$

$\delta_{\mu\nu}$ is the Kronecker symbol.

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